Floquet theory: a stroboscopic approach to study stability of a limit-cycle solution

Lingfa Yang*
Department of Chemistry and Volen Center for Complex Systems, MS 015,
Brandeis University, Waltham, Massachusetts 02454-9110, USA

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The reaction-diffusion systems take form:

$$\frac{\partial X(x,t)}{\partial t} = F(X) + \mathbf{D}\nabla^2 X \tag{1}$$

where $X = \{X_{i=1\ldots n}\}$ are concentration vectors; x is space vector; t is time. $F = \{F_{i=1\ldots n}\}$ are chemical kinetics vectors of the reaction. $\mathbf{D}$ is a diffusion coefficient matrix. The laplasian, $\nabla^2 = \frac{\partial^2}{\partial x^2}$(1D), or $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$(2D).

Let's assume that the system have a homogeneous limit-cycle $V' = F(V)$, or $V(t) = V(0) + \int_0^T F(X(t))dt$.

Linearization at the limit cycle gives

$$\frac{\partial Y}{\partial t} = \mathbf{A}(t)Y + \mathbf{D}\nabla^2 Y \tag{2}$$

where $\mathbf{A}(t) = \left|\frac{\partial F_i}{\partial X_j}\right|_{V(t)}$.

Solutions to the linearized problem are of the form

$$Y(x,t) = Q(t)e^{ikx} \tag{3}$$

where k is the wavenumber. Put back,

$$\frac{dQ}{dt} = (\mathbf{A}(t) - k^2\mathbf{D})Q(t) = \mathbf{B}(t)Q(t) \tag{4}$$

where Q is a vector with n components, and

$$\mathbf{B}(t) = \mathbf{B}(t+T) \tag{5}$$

is a T-periodic, $n \times n$ matrix, which arises from the linearization.

A *fundamental solution matrix* is any matrix whose columns are the components of linearly independent solutions of $\frac{dQ}{dt} = \mathbf{B}(t)Q$, Suppose

$$\mathbf{Q}(t) = [Q_1, Q_2, \ldots Q_n] \tag{6}$$

is a fundamental solution matrix. Then

$$\mathbf{Q}(t+T) = \mathbf{B}(t+T)\mathbf{Q}(t+T) = \mathbf{B}(t)\mathbf{Q}(t+T) \tag{7}$$

so $\mathbf{Q}(t+T)$ is a fundamental solution matrix if $\mathbf{Q}(t)$ is. It follows that we may express $\mathbf{Q}(t+T)$ as linear combinations of the columns of $\mathbf{Q}(t)$. Hence

$$\mathbf{Q}(t+T) = \mathbf{Q}(t) \cdot \mathbf{C} \tag{8}$$

where $\mathbf{C}$ is a $n \times n$ constant matrix which depends in fact on $\mathbf{Q}(0)$, and is of course a function of $\mathbf{B}(t)$

Let $\mathbf{\Phi}(t)$ be a fundamental solution matrix with initial value equal to the unit matrix:

$$\mathbf{\Phi}(0) = \mathbf{I}. \tag{9}$$

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*Electronic address: `lingfa@brandeis.edu`; URL: `http://hopf.chem.brandeis.edu/yanglingfa/`

Then $\mathbf{\Phi}(t+T) = \mathbf{\Phi}(t)\cdot\mathbf{C}$ so that when t=0

$$\mathbf{\Phi}(T) = \mathbf{C}. \tag{10}$$

The *monodromy matrix* is the value at $t = T$ of the fundamental solution matrix $\mathbf{Q}(t)$ satisfying $\mathbf{Q}(t+T) = \mathbf{B}(t)\mathbf{Q}(t+T)$ when $\mathbf{Q}(0) = \mathbf{I}$. So $\mathbf{Q}(t+T) = \mathbf{Q}(t)\cdot\mathbf{C}$ can be written as

$$\mathbf{\Phi}(t+T) = \mathbf{\Phi}(t)\mathbf{\Phi}(T) \tag{11}$$

$$\mathbf{\Phi}(2T) = \mathbf{\Phi}^2(T) \tag{12}$$

$$\mathbf{\Phi}(3T) = \mathbf{\Phi}(2T)\mathbf{\Phi}(T) = \mathbf{\Phi}^3(T) \tag{13}$$

$$\mathbf{\Phi}(nT) = \mathbf{\Phi}^n(T) \tag{14}$$

The eigenvalues of $\mathbf{\Phi}(T)$ are the *Floquet multipliers*. We find that

$$\mathbf{\Phi}(T)\cdot\mathbf{\Psi} = \lambda(T)\mathbf{\Psi} \tag{15}$$

$$\mathbf{\Phi}(nT)\cdot\mathbf{\Psi} = \lambda(nT)\mathbf{\Psi} \tag{16}$$

$$\mathbf{\Phi}^n(T)\cdot\mathbf{\Psi} = \lambda^n(T)\mathbf{\Psi} \tag{17}$$

Since $\mathbf{\Phi}(nT) = \mathbf{\Phi}^n(T)$ we have $\lambda^n(T) = \lambda(nT)$ so that we may define a *Floquet exponent* $\sigma = \xi + i\eta$ through the relation

$$\lambda(T) = e^{\sigma T} \tag{18}$$

and write the eigenvalue problem as

$$\mathbf{\Phi}(T)\cdot\mathbf{\Psi} = e^{\sigma T}\mathbf{\Psi} \tag{19}$$

Now we derive an eigenvalue problem for the exponent. First we define

$$Q(t) = \mathbf{\Phi}(t)\cdot\mathbf{\Psi} \tag{20}$$

It then follows that $Q(0) = \mathbf{\Psi}$ and

$$Q(t+T) = \mathbf{\Phi}(t+T)\cdot\mathbf{\Psi} = \mathbf{\Phi}(t)\mathbf{\Phi}(T)\cdot\mathbf{\Psi} = e^{\sigma T}\mathbf{\Phi}(t)\cdot\mathbf{\Psi} = e^{\sigma T}Q(t) \tag{21}$$

and

$$\frac{dQ}{dt} = \mathbf{B}(t)Q \tag{22}$$

Define a reverse function

$$P(t) = e^{-\sigma t}Q(t) \tag{23}$$

so that $P(t)$ is T-periodic because

$$P(t+T) = e^{-\sigma(t+T)}Q(t+T) = e^{-\sigma(t+T)}\left[e^{\sigma T}Q(t)\right] = e^{-\sigma t}Q(t) = P(t) \tag{24}$$

and an eigenvalue problem for the Floquet exponents are obtained

$$\frac{dP}{dt} = -\sigma P + \frac{dQ}{dt}e^{-\sigma t} \tag{25}$$

or

$$\left[B(t) - \frac{d}{dt}\right]P = \sigma P \tag{26}$$

Notice that the characteristic exponents are not uniquely defined, but the multipliers are. The multipliers are eigenvalues of the monodromy matrix.

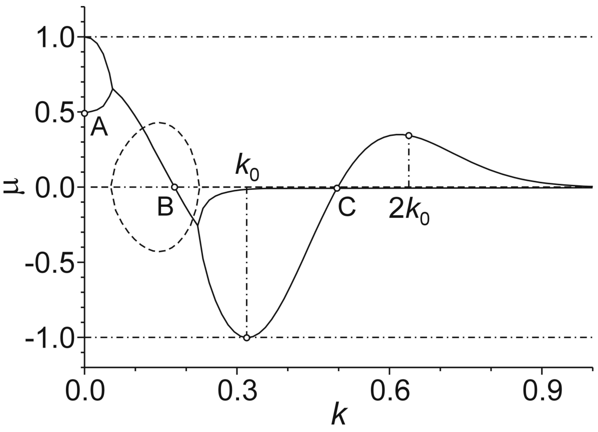

FIG. 1: Floquet multipliers $\mu_{1,2}$ (solid for real, dashed for imaginary part) of the limit cycle solution as a function of wavenumber k. [in *Phys. Rev. Lett.*, **92**, 198303 (2004).]

Although Floquet multipliers $\mu_j = e^{\sigma_j T}, j = 1, 2, \ldots, n$ are determined by $\mathbf{B}(t)$, it is not obvious how to calculate them, and the eigenvalues of $\mathbf{B}(t)$ do not seem to be extremely relevant. Their relations:

$$\Pi_{j=1}^{n} \mu_j = \exp\left(\int_0^T tr\mathbf{B}(t)dt\right) \tag{27}$$

or

$$\Sigma_{j=1}^{n} \sigma_j \equiv \frac{1}{T}\int_0^T tr\mathbf{B}(t)dt \left(mod\frac{2\pi i}{T}\right) \tag{28}$$

Its proof concerns LEMMA: If $\mathbf{Q}$ is an $n \times n$ matrix solution of the linear equation $\frac{dQ}{dt} = \mathbf{B}(t)Q$, then $\det \mathbf{Q}(t) = [\det \mathbf{Q}(t_0)] \exp\left(\int_0^T tr\mathbf{B}(t)dt\right)$ for all t, t_0.

Practical steps:

1) Calculate the period T precisely.
2) Add perturbation ΔX of wavenumber k to one component at a bulk state.
3) Integrate the 1D PDE for a period T (Poincare map, $\Phi(T)X(0) \to X(T)$).
4) Find deviation of each vector (one column), $[X(T) - X(0)]/\Delta X$.
5) Repeat 2,3,4, get the monodromy matrix $\mathbf{\Phi}$
6) Find eigenvalues of $\mathbf{\Phi}$, which are the expected multipliers μ as in FIG 1.