Linear stability of a 2-variable RD Model

$$\begin{cases} \frac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \frac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

$$J = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}_{ss} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{ss}, J_M = J - k^2 D$$

$$|J_M - \lambda I| = \begin{vmatrix} a_{11} - k^2 D_u - \lambda & a_{12} \\ a_{21} & a_{22} - k^2 D_v - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - Tr\lambda + \det = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left(Tr \pm \sqrt{\Delta}\right), \Delta = Tr^2 - 4\det$$

Brussllator: $\begin{cases} f(u,v) = a - (b+1)u + u^2 v \\ g(u,v) = bu - u^2 v \end{cases}$,

$$J = \begin{bmatrix} -(b+1) + 2uv & u^2 \\ b - 2uv & -u^2 \end{bmatrix}_{ss=(a,\frac{b}{a})} = \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix}$$

$$J_M = \begin{bmatrix} b - 1 + D_u \nabla^2 & a^2 \\ -b & -a^2 + D_v \nabla^2 \end{bmatrix}$$

$Tr = b - 1 - a^2 - k^2(D_u + D_v)$,

$\det = D_u D_v k^4 + \left[(1-b) D_v + a^2 D_u\right] k^2 + a^2$

The Hopf bifurcation (at $k = 0, Re\lambda = 0, Im\lambda = \omega$)

$Tr = 0$ gives $b_c^H = 1 + a^2$,and $\lambda_{1,2} = \pm ia$.

Oscillation exists for $b > b_c^H$.

The Turing bifurcation (at $k \neq 0, \lambda = 0, \frac{d\lambda}{dk} = 0$)

$\lambda_2 = 0$ needs $\det = 0$, which results in $b = k^2 D_u + \left(1 + \frac{D_u}{D_v} a^2\right) + \frac{a^2}{k^2 D_v}$

Let $\frac{db(k^2)}{dk^2} = 0$,which gives $k_c^2 = \frac{a}{\sqrt{D_u D_v}}$,and $b_c^T = \left(1 + a\sqrt{\frac{D_u}{D_v}}\right)^2$