Jumping solitary waves in an autonomous reaction–diffusion system with subcritical wave instability

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We describe a new type of solitary waves, which propagate in such a manner that the pulse periodically disappears from its original position and reemerges at a fixed distance. We find such jumping waves as solutions to a reaction–diffusion system with a subcritical short-wavelength instability. We demonstrate closely related solitary wave solutions in the quintic complex Ginzburg–Landau equation. We study the characteristics of and interactions between these solitary waves and the dynamics of related wave trains and standing waves.

Traveling waves in reaction–diffusion systems, particularly the Belousov–Zhabotinsky (BZ) reaction, have attracted a great deal of attention, for the insights they provide into pattern formation in chemical and biological systems.[1] In most cases, such waves propagate at a constant velocity. Solitary waves, *e.g.*, solitons or pulses in reaction–diffusion systems and cables,[2,3] have been studied in great detail because of their role in information transmission in natural and man-made systems.[3,4] Solitons are of particular interest owing to their particle-like behavior.[5,6] Recently, localized structures have drawn attention as a result of their potential importance in structureless memory devices.[7–10] In the simplest cases, solitary localized pulses are stationary in time and space or propagate smoothly with constant shape and velocity. In other instances, however, they display oscillations in amplitude or propagation speed.[11–15] Both stationary and oscillatory localized structures were recently observed in the BZ reaction in a reverse microemulsion.[16] It is known that localized structures can be found in extended systems with a subcritical oscillatory instability.[17] Here we present a new type of solitary traveling waves, which we call jumping oscillons (JO). They propagate in such a manner that the pulse periodically disappears and then reemerges at a fixed distance from its previous position. Thus, in a co-moving frame, the wave looks like a stationary oscillon. We find JO as solutions of an autonomous reaction–diffusion system with a subcritical short wavelength instability. We also obtain closely related solitary wave solutions in the quintic complex Ginzburg–Landau (GL) equation.

We consider a set of reaction–diffusion equations proposed by Purwins and co-workers, who found localized structures and solitons in this model,[18] which consists of the well known FitzHugh–Nagumo equations supplemented with a third variable.

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u\nabla^2 u + k_1 + 2u - u^3 - k_3 v - k_4 w,\\ \frac{\partial v}{\partial t} &= D_v\nabla^2 v + \frac{1}{\tau}(u - v),\\ \frac{\partial w}{\partial t} &= D_w\nabla^2 w + (u - w).\end{aligned} \quad (1)$$

Here u is the activator, and v and w are inhibitors with slow and fast diffusion, respectively, $D_v \ll D_w$. Phenomenologically, *e.g.*, in gas-discharge systems,[18] the activator can be thought of as the charge-carrier density in the discharge gap, which may grow autocatalytically, while the slow inhibitor represents the additional field built up by these charges in response to the applied voltage, and the fast inhibitor mimics an applied feedback coupling. For numerical simulations we employ an explicit Euler method with a spatial discretization of $\Delta x = 0.5$ space units (s.u.) per pixel and a time step $\Delta t = 0.001$ time units (t.u.).

We observe JO in eqn (1) with a subcritical wave instability, where a local excitation grows from the spatially uniform stable steady state (SS). Linear stability analysis of the SS solution, using a locally written software package to obtain the eigenvalues of the Jacobian matrix, reveals a parametric domain where the Hopf, Turing and wave bifurcation surfaces are situated close to one another. Fig. 1(a) shows these bifurcations in a section of the (k_1, k_4) plane. In every case the SS is stable below the lines and unstable above them as indicated in the diagram. We find JO in the region around point P, where the wave instability is subcritical [Fig. 1(b)]. In this region, JO, trains of JO, or standing waves (SW) can be induced, depending on the initial conditions. To the right of the dot-dashed line, standing waves are the only stable non-trivial (*i.e.*, non-constant) solutions. Thus, values of k_1 between the singularity point at about −8.8 and the wave bifurcation point at about −7.4 allow for multistability among SS, SW, JO and various trains of JO in the broad range to the left of the vertical dot-dashed line between about −8.8 and −7.5, and bistability between SS and SW in the narrow region between that line and the bifurcation point.[19]

Fig. 1(b) shows the abrupt transitions between the stable solutions, which are manifested in almost vertical segments of the unstable branch due to stiffness of the system (τ = 50). The analogous diagram for the soft system (2) is presented in Fig. 1(c).

Fig. 2(a) shows a single JO emerging in a system with zero-flux boundary conditions from the edge of a region in which u

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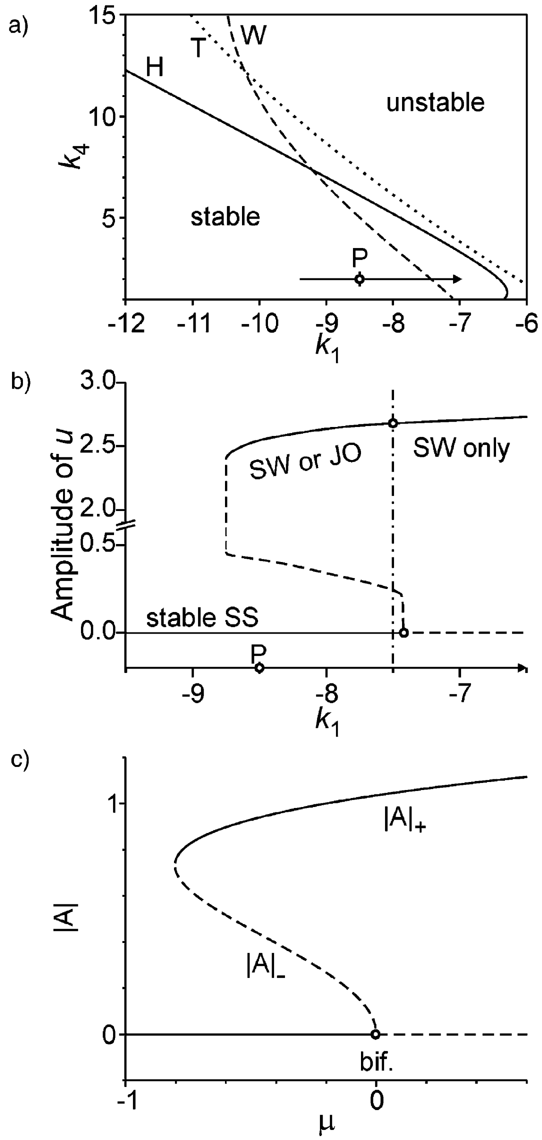

Fig. 1 Bifurcation diagram (a) Hopf (H), Turing (T) and wave (W) bifurcation lines. Parameters: $k_3 = 10$, $\tau = 50$; $D_u = D_v = 1.0$, $D_w = 60$. (b) Subcritical wave instability curve calculated with control parameter k_1 varied at fixed $k_4 = 2$ along the arrow in (a). (c) A more typical subcritical bifurcation in the quintic GL eqn (2) with parameters given in Fig. 4.

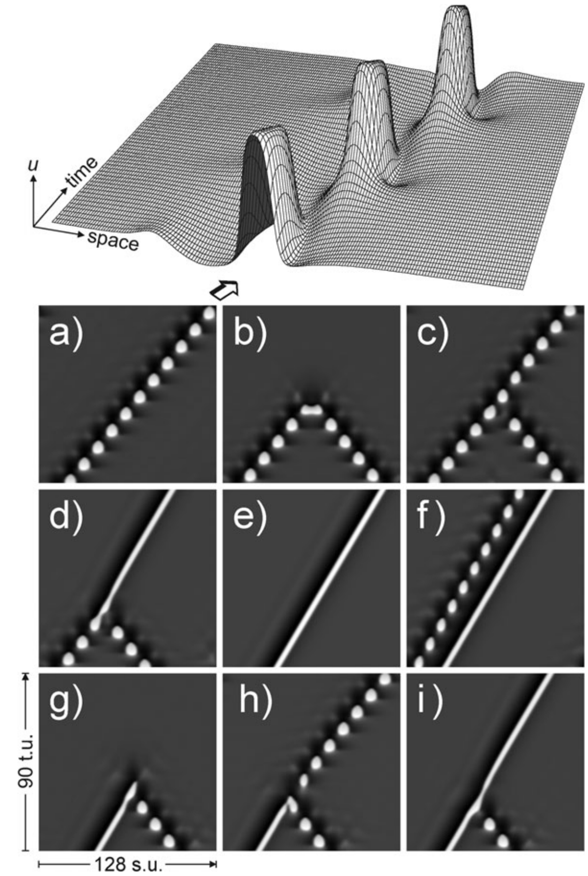

Fig. 2 Jumping oscillons and their interactions in model (1). Parameters correspond to P ($k_1 = -8.5$, $k_4 = 2.0$) in Fig. 1(a). Spatio-temporal plots (a)–(i) show component u as a gray level linearly related to u: black for $u = -1.2$, and white for $u = 1.2$. (a) A single JO; (b–d) collisions between JO; (e) a solitary propagating wave (SPW); (f) coexistence between a JO and an SPW with parameters the same as above, except $k_1 = -8.0$, and $D_v = 0.5$; (g–i) collisions between a JO and an SPW.

is perturbed by a superthreshold deviation from its SS value. At $t = 0$ most of the system is at the SS, $(u,v,w) = (u_{ss}, v_{ss}, w_{ss})$, where $u_{ss} = v_{ss} = w_{ss} = s = A + B$, and $A, B = (-q/2 \pm\sqrt{\Delta})^{1/3}$, $\Delta = (q/2)^2 + (p/3)^3$, $p = k_3 + k_4 - 2$, and $q = -k_1$, while in the perturbed region, $x \in [0,30]$ s.u., u is set to -0.2. The perturbation amplitude ($\Delta u = 0.6$) slightly exceeds the threshold, 0.4 [see Fig. 1(b)]. Initially, most of the perturbed region returns to the SS, except at the right end, where a localized JO appears and begins to propagate to the right.[20] It propagates in such a manner that the pulse periodically disappears from its original position and reemerges at a fixed distance, about 51 s.u. for the conditions in the figure.

When two JO collide with the same phase, they annihilate each other as shown in Fig. 2(b). With other phase relationships, one JO can die, leaving another to continue jumping [Fig. 2(c)]. In yet another case, collision of JO results in creation of a solitary propagating wave (SPW) with a constant amplitude [Fig. 2(d). A single SPW can be created from the same initial conditions as in Fig. 2(a) if we choose parameters corresponding to lower diffusivity ($D_v = 0.1$) and lower excitability ($k_4 = 1$).[20] A single JO moves faster than a SPW, as seen from the slopes of the traces in Fig. 2(a) and (e). If, after a SPW emerges, the system parameters are reset to values in the bistable region, the SPW can survive and coexist with the JO in a large parametric domain. If we start the SPW behind the JO, the SPW will lag further and further behind (not shown). Alternatively, when the JO chases the SPW, it catches up, and they eventually form a bound pair, maintaining a constant distance and propagating at the same velocity [Fig. 2(f)]. When a JO and a SPW collide they can annihilate each other [Fig. 2(g)], leave a single JO [Fig. 2(h)], or single SPW [Fig. 2(i)], depending on their phase relations.

We also studied the behavior of sequences of JO. If we make an initial perturbation in an interval at the boundary, *e.g.*, $u = -0.3$ for $x \in [0,3]$ s.u. with SS conditions everywhere else, a stable pacemaker emerges, which periodically generates JO. Inspection of the propagating sequence of JO in Fig. 3(a) shows that its wavelength is equal to two jumps of the constituent JO. This wavelength also corresponds to a double jump of a solitary JO. Therefore, it is convenient to take it as our reference wavelength λ_w, which equals 20.21 s.u. with the parameters shown in Fig. 1(a), point P.

In a system with zero-flux boundary conditions, a single JO disappears when it reaches or closely approaches the boundary. However, sequences of JO generated by pacemakers behave quite differently. If the system length is close to a whole number of reference half-wavelengths, $L = (n/2 \pm 0.20)\lambda_w$, SW containing an integral number of half-waves are set up [Fig. 3(a)]. If the system length is near a whole number of reference quarter-wavelengths, $L = [(n/2 + 1/4) \pm 0.05]\lambda_w$, interaction of the leading JO with the boundary creates conditions that prevent the final jump of the second JO, thereby initiating a back-propagating wave of annihilation, which brings the system to the SS [Fig. 3(b)].

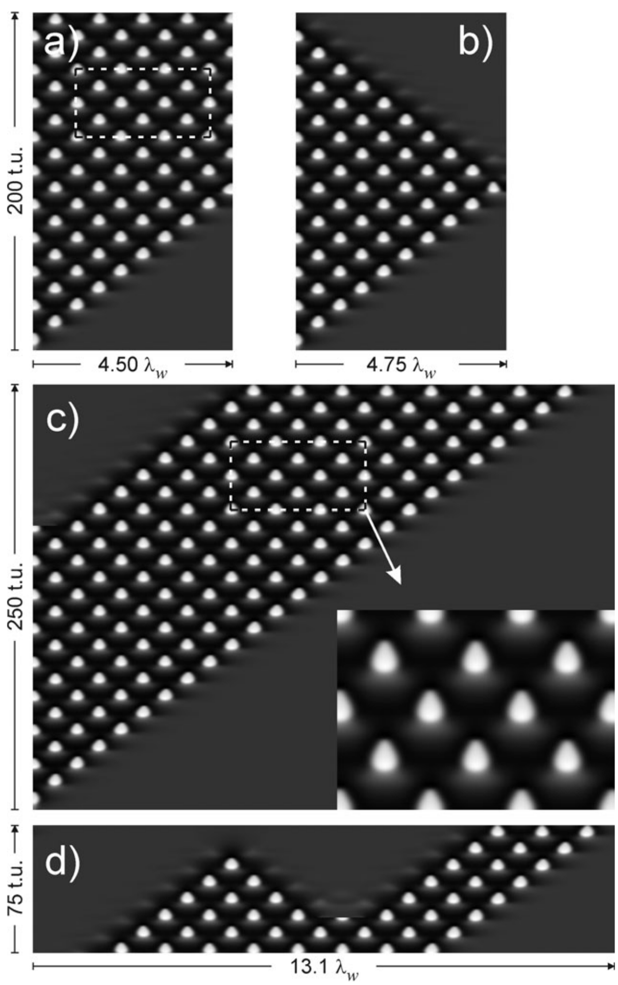

Fig. 3 Dynamics of sequences and trains of JO in the multistability region. (a) An infinite sequence of JO forms standing waves when $L_{SW} = (n/2 \pm 0.2)\lambda_w$. (b) Interaction of the first JO in a sequence with a zero-flux boundary at $L_x = [(n/2 + 1/4) \pm 0.05]\lambda_w$ triggers a back-propagating annihilation wave that eventually returns the system to its steady state. (c) A train of eight JO is created when the 9th JO is suppressed (see text). The dashed line surrounds a window with a pattern of standing waves indistinguishable from that in (a). (d) Continuation of the sequence in (c). Elimination of the 4th JO in the train at 30 t.u. triggers a back-propagating annihilation wave.

To clarify the peculiar dynamics of sequences of JO we also studied the behavior of finite trains of JO. A train of JO can be truncated at any desired number by setting the concentrations in the pacemaker area to their SS values at the moment when a new oscillon would be generated [Fig. 3(c)]. We find that a finite train of any length disappears after reaching the other boundary (not shown). Comparison of Fig. 3(a) and (c) demonstrates that it is practically impossible to distinguish stationary SW generated by a permanent pacemaker from the pattern of a propagating train within a finite space-time frame indicated by the dashed line. We further find that elimination of a single JO in a train (again, by setting concentrations to SS values where the JO would have appeared) triggers a back-propagating annihilation wave, which is a mirror image of the train tail [Fig. 3(d)]. Also, elimination of a single JO inside the global pattern of antiphase oscillations, which would otherwise fill the entire system [Fig. 3(a)], generates two waves that propagate in opposite directions and return the system to the SS (not shown).

On the other hand, in the region to the right of the dash-dotted line in Fig. 1(b) any supercritical perturbation of the SS results in establishment of robust SW, which occupy the entire system. This behavior differs from that of soft systems with a subcritical wave instability, where localized standing waves are stable.[21]

Symmetry considerations suggest that all the above results should also be obtained on a ring with twice the length of the line with zero-flux boundaries. In this case, a pacemaker generates two sequences of JO, which propagate in opposite directions and collide at the other side of the ring. We have obtained symmetrically doubled versions of the patterns shown in Fig. 3 in ring systems with doubled lengths (not shown). On the other hand, ring systems permit the study of infinite propagation of solitary waves and wave trains generated by asymmetric initial conditions.[22] We find that trains of JO maintain constant length during propagation (not shown), which points to anomalous dispersion.[23]

To look for analogous solutions in a more general context, we turn to the quintic complex Ginzburg–Landau equation:[17]

$$\frac{\partial A}{\partial t} = \mu A + \alpha|A|^2 A + \beta\nabla^2 A + \eta|A|^4 A \qquad (2)$$

Eqn (2) describes a subcritical Hopf bifurcation at $\mu_r = 0$ (the subscript r indicates the real part) with three branches: $|A| = 0$, and $|A|_{\pm} = \sqrt{\frac{-\alpha_r \pm \sqrt{\alpha_r^2 - 4\eta_r\mu_r}}{2\eta_r}}$. The first branch is stable (unstable) for negative (positive) μ_r; the second exists only for $\alpha_r^2/(4\eta_r) < \mu_r < 0$, and is always unstable; the last exists for $\mu_r > \alpha_r^2/(4\eta_r)$, and is always stable.

Akhmediev *et al.*[14,15] have found a variety of localized structures and propagating waves with non-trivial dynamics in an equivalent system. Here we find that a solitary propagating oscillon (PO) shown in Fig. 4(a) is a solution of eqn (2). An ordinary oscillon is a localized stationary structure. It can be triggered by a superthreshold perturbation of the trivial zero-amplitude steady state background.[20] When the symmetry of

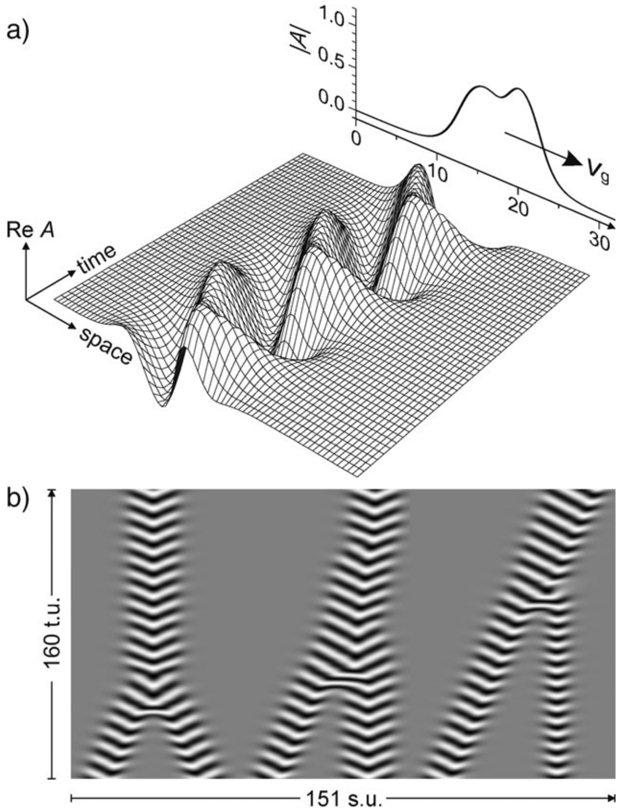

Fig. 4 Propagating oscillons in the complex GL eqn (2) with parameters $\mu = -0.1 \pm 0.1i$, $\alpha = 3.0 + i$, $\eta = -2.8$, and $\beta = 1.0 + i$. (a) Phase oscillation and modulus profile. (b) Collisions involving JO and stationary oscillons shown by gray level spatio-temporal plots, where the gray level is linearly proportional to the real part of the amplitude, with black corresponding to –1, and white to 1.

such an oscillon is broken by a sufficiently strong perturbation, it begins to propagate, becoming a PO.[20] The modulus of the PO possesses a constant profile and speed, while its phase propagates *via* jump-like oscillations.

We studied the interaction of these POs with each other and with stationary oscillons, which are also stable solutions of eqn (2).[17] Fig. 4(b) shows three scenarios. Collision of two POs results in emergence of a broad stationary oscillon. Collision of a PO with such an oscillon causes the disappearance of the PO and a slight shift in the position of the stationary oscillon. If a PO collides with a narrow oscillon, the PO annexes the oscillon, increasing its width.

We have reported a new type of solitary wave, which we call a jumping oscillon. This phenomenon combines features of both solitons and oscillons: constant motion and sustained oscillation. It can be generated in pure reaction–diffusion systems with a subcritical wave instability. We have examined the coexistence between different wave types, their competition and collision. The existence of JO requires a high value of D_w in comparison with the two other diffusion coefficients. This condition can be fulfilled in the Belousov–Zhabotinsky reaction in sodium bis(2-ethylhexyl) sulfosuccinate microemulsion (BZ–AOT system),[24] where nanometer-diameter water droplets carrying activator and inhibitor species diffuse much more slowly than the inhibitor species in the oil phase. The model explored here is close to a simplified model of this system,[24] which suggests that the BZ–AOT system may be a promising candidate to generate JO in experiments. We have used the model developed by the Purwins group[18] in its original form, where variables can take negative values during parts of the cycle. The model can be converted into chemical (or concentration) form without changing its dynamics by shifting the variables into the positive octant of the phase space and replacing the negative zero-order reaction term k_1 by an explicit Langmuir expression with a very small Langmuir constant. By virtue of containing both frequency and phase information, JO may have advantages over solitons in communication applications.

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