Pattern formation arising from wave instability in a simple reaction-diffusion system

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Pattern formation is studied numerically in a three-variable reaction-diffusion model with onset of the oscillatory instability at a finite wavelength. Traveling and standing waves, asymmetric standing-traveling wave patterns, and target patterns are found. With increasing overcriticality or system length, basins of attraction of more symmetric patterns shrink, while less symmetric patterns become stable. Interaction of a defect with an impermeable boundary results in displacement of the defect. Fusion and splitting of defects are observed. © 1995 American Institute of Physics.

I. INTRODUCTION

Pattern formation in reaction-diffusion systems has been a subject of intensive study for the past 40 years.1–6 The simplest patterns are periodic variations of concentrations. They can be purely spatial, purely temporal or spatiotemporal. The simplest supercritical bifurcations of a spatially uniform steady state result in spontaneous formation of these fundamental patterns,7,8 The space-independent Hopf bifurcation (SIHB) gives rise to concentration oscillations that are uniform in space and periodic in time. The Turing bifurcation leads to the emergence of concentration profiles that are stationary in time and periodic in space.

Along with the famous instability that is associated with his name, Turing also discovered an oscillatory instability with finite wavelength that results in “genuine traveling waves.” He pointed that one needs at least three species to obtain this instability in a reaction-diffusion system with a diagonal diffusion matrix.

If we consider a three-species reaction-diffusion system in one spatial dimension and substitute the Fourier modes

\[ u = U \exp(\alpha x + ikx) \]

into the corresponding linearized partial differential equations, then the characteristic equation takes the form

\[ \lambda^3 - T \lambda^2 + F \lambda - J = 0, \]

where \( T \) is the trace of the corresponding Jacobian matrix, \( F \) is the sum of all the second order determinant of the Jacobian matrix, and \( J \) is the Jacobian. We are interested in the case with one real negative and a complex conjugate pair of eigenvalues,

\[ p_1 < 0, \quad p_{2,3} = \alpha \pm i \omega. \]

The wave instability (WI) takes place when \( \alpha > 0 \) at \( k > 0 \), and \( \alpha < 0 \) at \( k = 0 \).

Some attempts have been made to analyze the basic spatiotemporal patterns: traveling (TW) and standing waves (SW) and target patterns: (TP), which can emerge in reaction-diffusion systems in the vicinity of a supercritical wave bifurcation (WB).9,10 Recently Levine and Zoe found the WB in a semiphenomenological model of CO oxidation on Pt which gives rise to SW, in qualitative agreement with experiment.11

To date, however, there has been no systematic study of the wave instability region in any reaction-diffusion model. More importantly, a convincing demonstration of target pattern formation due to WI has not yet been reported.

Understanding the mechanism of the TP formation is important because spontaneous desynchronization of bulk oscillations in experimental reaction-diffusion systems usually begins with the emergence of TP.12–14 In many cases TP emerge from pacemakers formed by local inhomogeneities in the system parameters.14–16 On the other hand, there is a good deal of evidence that TP can arise from fluctuations in the concentrations, which are the dynamic variables of the system.13,17–20

Target patterns have been found to emerge due to large amplitude perturbations of limit cycle bulk oscillations in some reaction-diffusion models.21,22 They also arise as solutions of the Ginsburg–Landau equations derived in the vicinity of the Hopf–Turing codimension-2 bifurcation.23–25 Still, the wave instability seems to be the most likely cause of TP formation arising from small concentration fluctuations.

Here we perform a numerical study of a simple reaction-diffusion model with a relatively large domain of the wave instability. We present various stationary spatiotemporal patterns as well as transient patterns.

II. MODEL

Because the conditions for emergence of the wave bifurcation are restrictive, we investigate a simple formal chemical model which permits easy manipulation of the wave instability domain. We employ here the following reaction scheme, related to the Brusselator and other models with so-called cubic autocatalysis as the main source of instability.1,26

\[ X + 2Y \rightarrow 3Y, \]
\[ S_1 + 2Z \rightarrow X + 2Z, \]
\[ Y \rightarrow P_1, \]
\[ S_2 + X \rightarrow X + Z, \]

(1) (R1) (R2) (R3) (R4)
Both Neumann ~315X and periodic boundary conditions are employed in the simulations.

The one-dimensional reaction-diffusion model corresponding to reaction scheme (R1)–(R7) is as follows:

\[
\begin{align*}
\frac{\partial X}{\partial t} &= -k_1XY^2 + k_2S_2Z^2 - \frac{k_6CX}{K_m + X} + D_x \frac{\partial^2 X}{\partial l^2}, \\
\frac{\partial Y}{\partial t} &= k_1XY^2 - k_3Y + k_7S_3 + D_y \frac{\partial^2 Y}{\partial l^2}, \\
\frac{\partial Z}{\partial t} &= k_4S_2X - k_5Z + D_z \frac{\partial^2 Z}{\partial l^2},
\end{align*}
\]

where \( K_m \) is the Michaelis constant of reaction (R6).

The corresponding nondimensional scaled model is

\[
\begin{align*}
\frac{\partial X}{\partial \tau} &= m \left( -x y^2 + z \frac{a x}{g + x} \right) + d_x \frac{\partial^2 x}{\partial r^2}, \\
\frac{\partial y}{\partial \tau} &= n(x y^2 - y + b) + d_y \frac{\partial^2 y}{\partial r^2}, \\
\frac{\partial z}{\partial \tau} &= x - z + d_z \frac{\partial^2 z}{\partial r^2},
\end{align*}
\]

where

\[
X = x_0x, \quad Y = y_0y, \quad Z = z_0z, \quad \tau = k_5t, \quad r^2 = \frac{k_5}{D_z} l^2.
\]

\[
x_0 = \frac{k_5^2 k_2}{k_1k_2k_5^2S_2}, \quad y_0 = \frac{k_3}{k_1x_0}, \quad z_0 = \frac{k_4S_2}{k_5} x_0, \quad d_x = \frac{D_x}{D_z}, \\
d_y = \frac{D_y}{D_z}, \quad m = \frac{k_3y_0}{k_5x_0}, \quad n = \frac{k_3}{k_5}, \quad a = \frac{k_6C}{k_3y_0}, \quad b = \frac{k_7S_3}{k_3y_0}, \\
g = \frac{K_m}{x_0}.
\]

In our simulations, the parameter \( g \) is always equal to \( 1 \times 10^{-4} \). If not otherwise indicated, the following parameters are also kept constant: \( a = 0.9, b = 0.2, n = 15.5, d_x = d_y = 0 \). The length of the system, \( L \), and \( m \) and \( n \) are variable parameters. Both Neumann (zero flux) and periodic boundary conditions are employed in the simulations.

III. METHODS OF SIMULATION

For numerical analysis of the space-independent system we employ the CONT numerical bifurcation and continuation package.4 Simulations of the one-dimensional reaction-diffusion system are done using a finite-difference approximation of Eq. (5). The corresponding system of ordinary differential equations is solved with the LSODE subroutine,28 using numerical estimates of Jacobian matrix.

The one-dimensional reaction-diffusion model corresponding to reaction scheme (R1)–(R7) is as follows:

\[
\frac{dX}{dt} = -k_1XY^2 + k_2S_2Z^2 - \frac{k_6CX}{K_m + X} + D_x \frac{\partial^2 X}{\partial l^2},
\]

\[
\frac{dY}{dt} = k_1XY^2 - k_3Y + k_7S_3 + D_y \frac{\partial^2 Y}{\partial l^2},
\]

\[
\frac{dZ}{dt} = k_4S_2X - k_5Z + D_z \frac{\partial^2 Z}{\partial l^2},
\]

where \( K_m \) is the Michaelis constant of reaction (R6).

The corresponding nondimensional scaled model is

\[
\begin{align*}
\frac{dx}{d\tau} &= m \left( -xy^2 + z \frac{ax}{g + x} \right) + dx \frac{\partial^2 x}{\partial r^2}, \\
\frac{dy}{d\tau} &= n(xy^2 - y + b) + dy \frac{\partial^2 y}{\partial r^2}, \\
\frac{dz}{d\tau} &= x - z + dz \frac{\partial^2 z}{\partial r^2},
\end{align*}
\]

where

\[
X = x_0x, \quad Y = y_0y, \quad Z = z_0z, \quad \tau = k_5t, \quad r^2 = \frac{k_5}{D_z} l^2.
\]

\[
x_0 = \frac{k_5^2 k_2}{k_1k_2k_5^2S_2}, \quad y_0 = \frac{k_3}{k_1x_0}, \quad z_0 = \frac{k_4S_2}{k_5} x_0, \quad d_x = \frac{D_x}{D_z}, \\
d_y = \frac{D_y}{D_z}, \quad m = \frac{k_3y_0}{k_5x_0}, \quad n = \frac{k_3}{k_5}, \quad a = \frac{k_6C}{k_3y_0}, \quad b = \frac{k_7S_3}{k_3y_0}, \\
g = \frac{K_m}{x_0}.
\]

In our simulations, the parameter \( g \) is always equal to \( 1 \times 10^{-4} \). If not otherwise indicated, the following parameters are also kept constant: \( a = 0.9, b = 0.2, n = 15.5, d_x = d_y = 0 \). The length of the system, \( L \), and \( m \) and \( n \) are variable parameters. Both Neumann (zero flux) and periodic boundary conditions are employed in the simulations.

IV. RESULTS

A. Local dynamics

Model (5) has three spatially uniform non-negative steady states (SS). Figure 1 shows the dependence of the stationary concentrations, \( x_0 = z_0 \) and \( y_0 \), on \( a \) and \( b \). The topmost state SS1 has one real negative and a pair of complex conjugate eigenvalues; SS2 has two negative and one positive real eigenvalues; SS3 (\( x_0 = z_0 = 0, y_0 = b \)) has all eigenvalues real and negative. Numbers in figures show values of \( b \).

The error tolerances are \( 1 \times 10^{-8} \) relative and \( 1 \times 10^{-12} \) absolute. The number of gridpoints varies with the length of the system. For \( 2 < L < 30 \) we use 300 gridpoints; for longer systems we maintain a constant resolution of 10 points per space unit.

B. System behavior

Figure 2 shows the lines of space-independent saddle-node and Hopf bifurcations of SS1 in the \( a,b \)-plane. Figure 3 depicts the lines of SIHB and WB of SS1 in the \( m,n \)-plane for \( a = b = 0 \), and for \( a = 0.9, b = 0.2 \).

In Fig. 4 we show dispersion curves (real part \( \alpha \) and imaginary part \( \omega \) of the eigenvalues vs wave number \( k \)) for SS at several \( m \). The group velocity \( \partial \omega / \partial k \) is positive in the entire domain of interest. SS1 is stable if \( m > m_c = 28.57 \). In what follows, we will refer to \( \epsilon = (m_c - m)/m_c \) as the over-
criticality. The domain of pure wave instability corresponds to $26.798 < m < 28.57$. If $m < 26.797$, $SS_1$ is unstable to both spatially nonuniform and uniform small perturbations, with the increment of instability rising as $k$ increases from 0 to $k_{\text{max}}$, where $k_{\text{max}}$ is the wave number at the maximum of $a$. In other words, the amplitude of the corresponding wave mode increases faster than that of the bulk oscillation. The wave number $k_{\text{max}}$ is almost independent of $m$; at $m = 28.57$, $k_{\text{max}} = 1.834$. It decreases with decreasing $m$ to a minimum of 1.818 at $m = 20$; it then increases slightly, rising to $k_{\text{max}} = 1.820$ at $m = 18$.

**B. Transient processes**

In most of our simulations we employ as initial conditions the locally perturbed homogeneous steady state $SS_1$. In some cases we start from perturbations of the limit cycle homogeneous bulk oscillations. Local perturbations are generated by increasing the value of $y$, usually at one gridpoint, or occasionally at two or three adjacent gridpoints. The increase varies from $1 \times 10^{-4}$ to 0.2.

For periodic boundary conditions a symmetric perturbation consists of an increase of $y$ at one gridpoint. Asymmetric perturbation is achieved either by increasing $y$ at one point and decreasing $y$ at an adjacent point, or by different positive deviations from $SS_1$ at two separated gridpoints.

Asymmetric perturbations with zero flux boundary conditions are obtained by single point perturbations either at one wall or in the interior. Symmetric perturbations involve equal changes at both walls.

In most cases, different initial conditions result in the same final patterns. Instances in which bistability occurs are noted below.

A local small perturbation gives rise to a wave packet that propagates with the group velocity $\phi \alpha \lambda k$. The width of the packet increases as it propagates [Figs. 5(a) and 6]. In the parameters region adjacent to the wave bifurcation (small overcriticality), the instability is convective;\textsuperscript{29,30} in the comoving frame the deviations grow, while in the initial system of coordinates they damp behind the propagating wave packet and the system returns to the steady state. When the instability increment ($a$) increases, the damping behind the wave packet decreases. Eventually the instability becomes global.

With zero flux boundary conditions, after a wave packet collides with a wall, a reflected wave appears, forming a transient localized region of standing waves [Fig. 5(a)]. With every reflection the SW regions become wider and eventually they occupy the entire system. The next phase of the evolution involves adjustment of the SW to the system length and elimination of defects and dislocations. If successful, the process results in a uniform SW [Fig. 5(b)].

A parallel evolution of the pattern takes place under periodic boundary conditions with a symmetric local perturbation. In this case, a transient region of SW develops from the point of collision of counterpropagating waves. If the perturbation is asymmetric, the final stationary pattern is a traveling wave (Fig. 6).

**C. Stationary patterns**

We perform a global qualitative study of pattern formation in a parameter domain beyond the onset of the wave instability. Our goal is to find stable patterns with relatively large basins of attraction in the $\epsilon, L$-parameter plane for our standard set of parameters. We use several initial conditions for the chosen set of parameters to search for coexistence of different patterns. In several cases our simulations reveal a bistability, but we do not study these secondary bifurcations further in this paper.

1. **Periodic boundary conditions**

Very near the WB, both SW (Fig. 5) and TW (Fig. 6) are stable in a ring system. At $m = 28.5$, $L = 16.4$, both symmetric and asymmetric local perturbations result in SW with wavelength $\lambda = L/5$. Periodic initial conditions with this
wavelength result in TW. For $m = 28$, or 28.2, symmetric local perturbations result in SW, while asymmetric local perturbations result in TW.

At a large distance from the onset of wave instability ($m = 18$), asymmetric perturbations result in TW, and symmetric perturbation result in symmetric target patterns. Figure 7 shows an asymmetric view of a symmetric TP in order to highlight the wave emitting region without cutting either the leading center or the region of the wave collision. We show the envelope of the target patterns with an overlay of 45 consecutive concentration profiles at a time step of 0.02.

2. Zero flux boundary conditions

In Fig. 8 we present a structure diagram in the $eL$-plane for the system with zero flux boundary conditions. We employ a single gridpoint perturbation at one wall with amplitude 0.2. The simulations are run for 500 time units at each point.

The SW are stable at low overcriticality in relatively short systems. For instance, at $m = 28.3$, not only asymmetric local perturbation but also symmetric perturbation at both walls results in antisymmetric SW with the halfwave length.

With increasing system length ($L$) or overcriticality (decrease of $m$), the basin of attraction of SW shrinks, and asymmetric patterns appear. These patterns are intermediate between SW and TP; most of the pattern displays a strong dominance of one of the constituent traveling waves, which forms a TP [Fig. 9(a)], however, another wave has a significant amplitude everywhere [Fig. 9(b)]. We term these patterns standing-traveling waves (STW).

A further increase of overcriticality, $0.09 \leq e \leq 0.30$, causes STW to become aperiodic in time at the 500 unit time scale (Fig. 10). At $e=0.37$ target patterns are stable with $L$ between 3.5 and 58. Standing waves were never obtained for $e=0.37$ for either symmetric or asymmetric perturbations. We have checked the stability of standing waves by employing initial conditions corresponding to the profiles of standing waves. Simulations for $L=12$ and $L=14$ show that standing waves are unstable at $e=0.37$.

3. Target patterns

A target pattern is a spatial domain filled with traveling waves that originate from a small region having at least one point with $\partial \alpha / \partial r = 0$. This region has been termed a leading...
center (LC), homogeneous pacemaker, or source. The period of oscillation of a LC depends upon $L$, tending to a limit as $L$ increases (Table I).

A LC emerges as a result of a local perturbation (an initial defect) of a homogeneous state. The final position of a LC depends on the distance between the initial defect and a wall, and on $L$. When a single initial defect is placed at a wall and $L$ varies from 3.5 to 33.25, the LC is found at the same wall when $L=5.25$; 7.0; 8.75; 10.5; 14.0; 17.5; 28.0; 31.5; 33.25, and at the opposite wall when $L=3.5$; 21.0. When $L=35$, two LC result from a single defect; one LC is established at the same wall and the second LC stabilizes inside the system, at a distance $3\lambda$ from the opposite wall. The same configuration occurs as a long lasting transient when $L=33.25$, but the internal LC finally vanishes in this case. At relatively large $L(L=42, 43, 44, 56, 57, 58)$, only one LC emerges and becomes established at the same wall where the initial defect is placed.

We determined the dependence of the final position of the LC on the position of the initial defect at $L=43$. If the initial defect is placed near a wall, the final position of the LC is at the wall. If the wall-defect distance exceeds a threshold, the LC stabilizes further from the wall than the initial defect. When the initial distance is much larger than the wavelength, the position of LC coincides with that of the initial defect. Table II shows the dependence of the LC to wall distance on that of the initial defect to the wall.

D. Multistability

When the set of simulations presented in Fig. 8 is repeated with initial perturbations at both walls, the same results are obtained in most of the $eL$-diagram. However, bistability is found at a few points. For instance, at $m=27$, $L=17.5$ and $m=26$, $L=10.5$, symmetric initial conditions result in SW, while single perturbations result in STW.

A very different type of multistability can occur for confined systems. First, TW and SW with various wavelengths can match the length of the system. Also, a system can accommodate various numbers of target patterns. Both these multistabilities have been found in our simulations.

E. Effect of diffusion coefficients

We performed a limited set of simulations with a non-zero diffusion coefficient for the autocatalyst ($d_y$), or with both $d_x$ and $d_y$ positive. With $m=18$, $d_y=0$, $0.1\leq d_x\leq0.3$, $L=30$, a local perturbation results in a stationary TP with a
single LC. With $d_y = 0.4$ we obtain a nonstationary TP with a pair of LC, each of which slowly drifts in a limited region. Figure 12 shows this pattern on a short time scale. Nonstationary patterns with several LC emerge at $d_y > 0.5$.

With $d_x = d_y = 0.05$, the structure diagram in Fig. 8 remains qualitatively the same, with a slight overall shift owing to a decrease in the critical value $m_c$.

V. DISCUSSION

In our simulations the ratio of the system length to the pattern wavelength ranges from 0.3 to 18, corresponding to the range used in published experiments on chemical waves.4–6,14

We employ both periodic and zero flux boundary conditions in our simulations. Periodic boundary conditions are particularly useful in studying basins of attraction and stability of traveling waves and the development of a target pat-

FIG. 7. Development of a target pattern from symmetric local perturbation of a system with periodic boundary conditions. Parameters, $m = 18$, $L = 20$. (a) $r,t$-plot; I, transition period, $r$ from 0 to 30; II, stationary target pattern, $r$ from 60 to 90. (b) Stationary pattern, overlay of 45 consecutive profiles with $\delta r = 0.02$.

FIG. 8. Structure diagram in $\epsilon, L$-plane for a system with zero flux boundary conditions. $\epsilon$ is overcriticality, $L$ is system length. +, standing waves; ◆, standing-traveling waves; △, aperiodic standing-traveling waves; ■, target patterns. Dashed lines are drawn by eye to separate domains of different patterns.

FIG. 9. Stationary pattern of standing-traveling wave in a system with zero flux boundary conditions. Parameters, $m = 26$, $L = 16$. (a) $r,t$-plot, $r$ from 500 to 505. (b) Overlay of 40 consecutive profiles with $\delta r = 0.025$. 
terns. Zero flux boundary conditions are more appropriate for simulating experimental reaction-diffusion systems, and we use them in most of our simulations.

Our data show that with increasing overcriticality ($\epsilon$) or system length ($L$), basins of attraction of more symmetric patterns shrink, while less symmetric patterns become stable. In ring systems near the onset of wave instability, standing waves develop from both symmetric and asymmetric local defects, while to obtain TW it is necessary to take as initial conditions a periodic concentration profile which is close to a homogeneous bulk oscillations. Parameters, $m=18$, $L=40$; initial conditions, SS, with two gridpoints separated by $r=7.0$, where $y$ is increased by 0.2. (b) Splitting, a single defect gives rise to a pair of leading centers; parameters, $m=18$, $L=70$; initial conditions, SS, with two adjacent gridpoints where $y$ is increased by 0.2.

It is important to understand the transition from more symmetric to less symmetric patterns. One mechanism seems to be relevant at relatively large $L(L>25)$. For stationary patterns, the amplitude equations valid in the vicinity of the WB (Ref. 8) can be reduced to ordinary differential equations which govern the dependence of the envelope amplitudes on the space coordinate. Let $A$ be the real amplitude of the envelope for the rightward traveling wave and $B$ be that for the leftward one. In the $A,B$ phase plane steady states situated at the axes correspond to TW and those on the diagonal correspond to SW. It has been shown that stability of TW and SW depends on the ratio ($\gamma$) of the real part of the coefficient of the restrictive cubic term $|B|^2A$ to that of the term $|A|^2A$. If $\gamma>1$, the TW are stable; if $-1<\gamma<1$, the SW are stable. The results in Refs. 9, 31, and 32 were obtained for infinitely large $L$, where boundary conditions play no role. We intend to analyze the amplitude equations for our system, which has a finite length and specific boundary conditions, to check whether such a simple nonlinear effect is the sole or main cause of the parity-breaking transition.

Figure 8 summarizes the results of our simulations with zero flux boundary conditions. Near the onset of wave instability only the spatially uniform SS is stable if $L<\lambda_c/2$. With $L=\lambda_c/2$ the first SW emerges. Figure 4 shows that the matching conditions, $L=\left(\frac{n\lambda_c}{2}\right)$, become looser as the overcriticality increases. At $\epsilon=0.02$, SW is the only stable pattern at all $L$ tested. At $0.055<\epsilon<0.3$, with initial point perturbations, patterns appear in the following sequence as $L$ is increased; SW, STW, aperiodic (in time) STW. The transition from SW to STW is a bifurcation associated with a loss of parity. Aperiodic STW found close to the apparent boundary between domains of STW and nonperiodic STW in our 500

### Table I. Period of oscillation of target patterns in comparison with homogeneous bulk oscillations.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$L/\lambda$</th>
<th>Mode</th>
<th>$T$</th>
<th>$T/T_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>bulk oscillations</td>
<td>0.96</td>
<td>1</td>
</tr>
<tr>
<td>3.5</td>
<td>1</td>
<td>core of LC</td>
<td>0.82</td>
<td>0.85</td>
</tr>
<tr>
<td>5.15+3.6=8.75</td>
<td>2+1.4=3.4</td>
<td>TP (LC inside)</td>
<td>0.84</td>
<td>0.88</td>
</tr>
<tr>
<td>8.2</td>
<td>3</td>
<td>TP</td>
<td>0.91</td>
<td>0.95</td>
</tr>
<tr>
<td>8.75</td>
<td>3</td>
<td>TP</td>
<td>0.93</td>
<td>0.97</td>
</tr>
<tr>
<td>14</td>
<td>5</td>
<td>TP</td>
<td>0.91</td>
<td>0.95</td>
</tr>
<tr>
<td>16.4</td>
<td>6</td>
<td>TP</td>
<td>0.91</td>
<td>0.95</td>
</tr>
<tr>
<td>35</td>
<td>12</td>
<td>TP</td>
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<tr>
<td>57</td>
<td>21</td>
<td>TP</td>
<td>0.90</td>
<td>0.94</td>
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</table>

$aT_0$:period of homogeneous oscillations. Parameters, $m=18$; $n=15.5$; $a=0.9$; $b=0.2$; $g=1\times10^{-4}$; $d_x=d_y=0$.  

FIG. 10. Aperiodic standing-traveling waves in a system with zero flux boundary conditions. Each frame corresponds to $\tau$ from 400 to 460. Parameters (a) $m=22$, $L=7$; (b) $m=22$, $L=35$; (c) $m=22$, $L=21$; (d) $m=26$, $L=21$.  

FIG. 11. Fusion and splitting of initial defects in a system with periodic boundary conditions. (a) Fusion, two defects form a single leading center; parameters, $m=18$, $L=40$; initial conditions, SS, with two gridpoints separated by $r=7.0$, where $y$ is increased by 0.2. (b) Splitting, a single defect gives rise to a pair of leading centers; parameters, $m=18$, $L=70$; initial conditions, SS, with two adjacent gridpoints where $y$ is increased by 0.2.
FIG. 12. Target patterns in a system with zero flux boundary conditions. Parameters, $d_z=0$, $d_y=0.4$, $m=18$, $L=30$. (a) $r,t$-plot, $r$ from 1500 to 1505; (b) overlay of 45 consecutive profiles with $\Delta r=0.02$.

<table>
<thead>
<tr>
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<th>2.0</th>
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<th>7.0</th>
<th>10.0</th>
<th>11.0</th>
<th>12.0</th>
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<th>15.0</th>
<th>19.0</th>
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<tr>
<td>LC</td>
<td>0.0</td>
<td>16.0</td>
<td>0.0</td>
<td>0.0</td>
<td>10.0</td>
<td>6.1</td>
<td>9.9</td>
<td>10.0</td>
<td>11.6</td>
<td>12.8</td>
<td>14.0</td>
<td>15.0</td>
<td>19.0</td>
<td>21.0</td>
</tr>
</tbody>
</table>

*Parameters, $m=18$; $n=15.5$; $a=0.9$; $b=0.2$; $g=1 \times 10^{-4}$; $d_z=d_y=0$.

Target patterns consist of three regions; a leading center (source), a domain occupied by traveling waves, and a region adjacent to the point of collision of the waves with either a wall or an opposite wave train (sink). The sink region displays a strongly decaying standing wave. The sizes of the source and sink regions are practically independent of the system length.

It is interesting to compare our data with the results of modeling of wave patterns of convection in binary fluids. The envelope profile of our asymmetric target patterns with a LC at one wall and a sink at another is in good agreement with the filling structure found by Cross as a numerical solution of the Ginsburg–Landau equations derived in the vicinity of the WB. He found that the shock structure is stable, while the TP is unstable toward small displacements. In our system, both configurations can be either stable or unstable, depending on the system parameters. We also found a number of stable asymmetric patterns with one LC and two sinks or vice versa.

The basic patterns are robust with respect to initial conditions. We obtain same patterns when the amplitude of a single local perturbation of $SS_1$ is varied from $1 \times 10^{-3}$ to 0.2. We find the same LC with equal amplitude local perturbations of either $SS_1$ or the limit cycle homogeneous bulk oscillations.

On the other hand, our data suggest that interactions of defects in confined reaction-diffusion systems can be very complicated. The final distance of a stationary LC from a wall is a nonmonotonic function of the distance between the initial defect and the wall, if the latter is comparable with the wavelength. Only if the distance between the defect and the wall is large enough, does the LC remain at the location of the initial defect (Table II).

Interaction between defects and the evolution of solitary defects can be quite sensitive to initial conditions. The birth of a pair of LC from a single initial defect is a good example. An initial increment in $y$ of $1 \times 10^{-4}$ at a single gridpoint...
results in the birth of a pair of LC, while larger initial defects of amplitude 0.03 or 0.2 yield a single LC. A defect with amplitude 0.2 occupying two adjacent gridpoints, gives rise to a pair of LC, while a three gridpoint defect with the same amplitude does not.

The results presented here only begin to illustrate the wealth of behavior associated with the wave bifurcation in reaction-diffusion systems. Clearly, further modeling and experimental studies will be required to reveal the full range of spatiotemporal patterning of which such systems are capable.

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25 P. Borckmans, G. Dewel, A. De Wit, and D. Walgraef, in Ref. 6, p. 323.